

Exercise Sheet 4

MATH 598 / 784 • High-Dimensional Probability

For discussion Friday, October 2, 2026

Throughout, $C, c > 0$ denote absolute constants, whose values may change. Covering numbers use centers in the set being covered.

1. Sampling preserves covering numbers (Exercise 8.3.21)

Let F be a finite class of measurable functions on a probability space (Ω, μ) , with $|f| \leq 1$ for every $f \in F$. Fix $\varepsilon \in (0, 1)$ and write $M = N(F, L^2(\mu), \varepsilon)$. Assume $M \geq 2$.

1. Show that there exist points $x_1, \dots, x_m \in \Omega$, with repetitions allowed, such that the empirical measure $\mu_m = m^{-1} \sum_{k=1}^m \delta_{x_k}$ satisfies

$$N(F, L^2(\mu_m), \varepsilon/4) \geq M, \quad m \leq C\varepsilon^{-4} \log M.$$

2. Use Bernstein's inequality to improve the sample-size bound to $m \leq C\varepsilon^{-2} \log M$.

Hint. Choose M functions with pairwise $L^2(\mu)$ -distance greater than ε , then sample independently from μ . For each pair, consider $Z = (f(X) - g(X))^2$. For part 2, compare $\mathbb{E}Z^2$ with $\mathbb{E}Z$.

2. Mix before you sample

For n a power of two, define the normalized Walsh–Hadamard matrix recursively by

$$H_1 = (1), \quad H_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}.$$

Let $D = \text{diag}(\sigma_1, \dots, \sigma_n)$, where the σ_i are independent uniform signs in $\{-1, 1\}$.

1. Check that H_n is orthogonal and its entries are $\pm n^{-1/2}$. Find a unit vector x for which $H_n x$ has only one nonzero coordinate. Explain why applying H_n alone does not make coordinate sampling reliable for every fixed input vector.
2. For every fixed nonzero $v \in \mathbb{R}^n$ and $u \geq 0$, prove that

$$\mathbb{P} \left\{ \|H_n D v\|_\infty > \frac{u \|v\|_2}{\sqrt{n}} \right\} \leq 2ne^{-u^2/2}.$$

3. Let $T \subset \mathbb{R}^n$ contain $N \geq 2$ points, and fix $\varepsilon, \rho \in (0, 1)$. Independently of D , form I by retaining coordinates independently, each with probability $\delta \in (0, 1]$. Set $\Phi(x) = \delta^{-1/2} (H_n D x)_I$. Show that, with probability at least $1 - \rho$,

$$(1 - \varepsilon) \|x - y\|_2^2 \leq \|\Phi(x) - \Phi(y)\|_2^2 \leq (1 + \varepsilon) \|x - y\|_2^2 \quad \text{for all } x, y \in T,$$

provided

$$\delta n \geq C\varepsilon^{-2} \log(N/\rho) \log(nN/\rho).$$

If this sufficient bound exceeds n , retain all coordinates instead.

Hint. Apply part 2 to all pairwise differences, then condition on D and use the ℓ^∞ flatness version of the coordinate-sampling theorem from Lecture 7. The same signs and sampled coordinates must work for every pair.

3. Quadratic forms on nets (Exercise 4.4.3)

Let A be an $m \times n$ real matrix and $0 < \eta < 1/2$.

1. If $\mathcal{N}$ and $\mathcal{M}$ are finite η -nets of S^{n-1} and S^{m-1} , respectively, prove that

$$\max_{x \in \mathcal{N}, y \in \mathcal{M}} \langle Ax, y \rangle \leq \|A\|_{\text{op}} \leq \frac{1}{1 - 2\eta} \max_{x \in \mathcal{N}, y \in \mathcal{M}} \langle Ax, y \rangle.$$

2. If $m = n$ and A is symmetric, prove that

$$\max_{x \in \mathcal{N}} |\langle Ax, x \rangle| \leq \|A\|_{\text{op}} \leq \frac{1}{1 - 2\eta} \max_{x \in \mathcal{N}} |\langle Ax, x \rangle|.$$

Hint. For nearby unit vectors x, x_0 and y, y_0 , expand $\langle Ax, y \rangle - \langle Ax_0, y_0 \rangle$ by changing one argument at a time.

4. Deviations of the norm on a net (Exercise 4.4.4)

Let A be an $m \times n$ real matrix, let $\mu \geq 0$, and let $\mathcal{N}$ be a finite η -net of S^{n-1} , where $0 < \eta < 1/2$. Prove that

$$\sup_{x \in S^{n-1}} |\|Ax\|_2 - \mu| \leq \frac{C}{1 - 2\eta} \max_{x \in \mathcal{N}} |\|Ax\|_2 - \mu|.$$

Hint. Treat $\mu = 0$ separately and reduce the remaining case to $\mu = 1$. Apply Problem 3 to $A^\top A - I_n$. When converting back from squared norms to norms, distinguish whether the maximum deviation on the net is at most 1 or greater than 1.