

Exercise Sheet 3

MATH 598 / 784 • High-Dimensional Probability

For discussion Friday, September 25, 2026

1. The Orlicz norm

For $\alpha \geq 1$, define

$$\|Y\|_{\psi_\alpha} = \inf \left\{ r > 0 : \mathbb{E} \exp \left(\frac{|Y|^\alpha}{r^\alpha} \right) \leq 2 \right\}.$$

Show that $\|\cdot\|_{\psi_\alpha}$ is a norm on the vector space of real random variables on a fixed probability space for which it is finite, where random variables equal almost surely are identified.

2. Moment growth and tail bounds

Fix $\alpha > 0$. For a real random variable Y , define

$$\|Y\|_{\text{mom-}\alpha} = \sup_{m \geq 1} \frac{(\mathbb{E}|Y|^{2m})^{1/(2m)}}{m^{1/\alpha}},$$

where the supremum is over positive integers. Prove the following two implications, with constants independent of the distribution of Y .

1. For every $B > 0$, there exist $A > 1$ and $c > 0$, depending only on α and B , such that

$$\|Y\|_{\text{mom-}\alpha} \leq B \implies \mathbb{P}\{|Y| \geq t\} \leq Ae^{-ct^\alpha} \quad \text{for all } t \geq 0.$$

2. For every $A > 1$ and $c > 0$, there exists $B > 0$, depending only on α , A , and c , such that

$$\mathbb{P}\{|Y| \geq t\} \leq Ae^{-ct^\alpha} \quad \text{for all } t \geq 0 \implies \|Y\|_{\text{mom-}\alpha} \leq B.$$

3. Concentration of the Euclidean norm

Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ have independent, centered coordinates of variance 1. Suppose each coordinate is K -subgaussian, in the variance-proxy convention of the lectures:

$$\mathbb{E}e^{\lambda X_i} \leq e^{K\lambda^2/2} \quad \text{for every } \lambda \in \mathbb{R} \text{ and } 1 \leq i \leq n.$$

Use Bernstein's inequality to show that, for an absolute constant $C > 0$,

$$\left| \|X\|_2 - \sqrt{n} \right|_{\psi_2} \leq CK.$$

4. Gaussian width of sparse vectors

Let $1 \leq s \leq N$ be integers, and let

$$V_s = \{x \in \mathbb{R}^N : \|x\|_2 = 1, \#\{i : x_i \neq 0\} \leq s\}.$$

Recall that $w(V_s) = \mathbb{E} \sup_{x \in V_s} \langle G, x \rangle$, where $G \in \mathbb{R}^N$ has independent standard normal coordinates. Show that, for an absolute constant $C > 0$,

$$w(V_s)^2 \leq Cs \log \left(1 + \frac{N}{s} \right).$$