# Exercise Sheet 1

MATH 598 / 784 • High-Dimensional Probability

**For discussion Friday, September 11, 2026**

These problems are adapted from Section 4.2 of the first edition of Roman Vershynin's *High-Dimensional Probability*. The edition matters: the exercise numbers differ in the second edition.

Throughout, let $(T, d)$ be a metric space and let $K \subseteq T$. An $\varepsilon$-net of $K$ is a subset $\mathcal{N} \subseteq K$ such that every point of $K$ lies within distance $\varepsilon$ of some point of $\mathcal{N}$. The covering number $N(K, d, \varepsilon)$ is the smallest cardinality of such a net. A subset $\mathcal{P} \subseteq K$ is $\varepsilon$-separated if $d(x, y) > \varepsilon$ for all distinct $x, y \in \mathcal{P}$; the packing number $P(K, d, \varepsilon)$ is the largest cardinality of an $\varepsilon$-separated subset of $K$.

## 1. Packing balls into $K$ (Exercise 4.2.5)

1. Suppose that $T$ is a normed space, with the metric induced by its norm. Prove that $P(K, d, \varepsilon)$ is the largest number of pairwise disjoint closed balls of radius $\varepsilon/2$ whose centers lie in $K$.
2. Give an example showing that the preceding statement can fail in a general metric space.

## 2. Allowing centers outside $K$ (Exercise 4.2.9)

In the definition of $N(K, d, \varepsilon)$, the centers of the balls in a covering are required to lie in $K$. Define the exterior covering number $N^{\text{ext}}(K, d, \varepsilon)$ in the same way, except that the centers may lie anywhere in $T$. Prove that

$$N^{\text{ext}}(K, d, \varepsilon) \le N(K, d, \varepsilon) \le N^{\text{ext}}(K, d, \varepsilon/2).$$

## 3. Monotonicity (Exercise 4.2.10)

Give a counterexample to the assertion

$$L \subseteq K \quad \Longrightarrow \quad N(L, d, \varepsilon) \le N(K, d, \varepsilon).$$

Then prove the approximate monotonicity statement

$$L \subseteq K \quad \Longrightarrow \quad N(L, d, \varepsilon) \le N(K, d, \varepsilon/2).$$

## 4. The Hamming cube (Exercise 4.2.16)

For $x, y \in \{0, 1\}^n$, define the Hamming distance

$$d_H(x, y) = \#\big\{i : x_i \ne y_i\big\}.$$

Let $K = \{0, 1\}^n$. Prove that, for every integer $m \in \{0, \ldots, n\}$,

$$\frac{2^n}{\displaystyle\sum_{k=0}^{m} \binom{n}{k}} \le N(K, d_H, m) \le P(K, d_H, m) \le \frac{2^n}{\displaystyle\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}.$$

*Hint.* Adapt the volumetric argument, replacing volume by cardinality.

Source: Roman Vershynin, *High-Dimensional Probability: An Introduction with Applications in Data Science*, first edition, Cambridge University Press, 2018, Exercises 4.2.5, 4.2.9, 4.2.10, and 4.2.16. The author provides the first-edition PDF free of charge.